Lifetime Portfolio Selection: A Simple Derivation

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Abstract

Merton's portfolio problem involves finding the optimal asset allocation between a risky and a risk free asset, and the optimal consumption over time, so as to maximize aggregate utility of consumption. This paper gives a simple and straightforward solution to Merton's portfolio problem for lognormally distributed returns and isoelastic utility in the discrete and continous time cases.

1 Introduction

Given a risk free asset and a risky, or volatile asset, Merton's portfolio problem involves determine the asset allocation and consumption that will maximize lifetime utility. For lognormally distributed returns, constant relative risk aversion, and a fixed finite or infinite lifespan, Merton's portfolio problem has an analytical solution. This was shown by Merton (1969) in the continuous time case. Samuelson (1969) studied the discrete time case, and determined the optimal consumption in the discrete time case. However, the optimal asset allocation for the discrete time case was not explored analytically. Nor has the relationship between the discrete time case and the continuous time case been explored.

Despite the importance of the solution to Merton's portfolio problem to the field of finance, it is rarely covered in introductory finance texts such as Bodie et al. (2009), and is primarily studied within economics, not finance. Part of the reason for this may be because introductory finance courses normally deal with discrete rather than continuous time, and also because Merton solved the problem by making inspired guesses as to the solution of systems of simultaneous second order differential equations.

Consequently there is a need for a solution to Merton's portfolio problem in the discrete time case that is straightforward, and does not involving the simultaneous solution of system of second order differential equations. In addition there is a need to explore optimal asset allocation in the discrete time case, and to see how the discrete time case maps onto the continuous time case as the time interval goes to zero.

2 Preliminaries

2.1 The lognormal distribution

Let $\mathbf{Z}$ represent the return distribution. As a result of the central limit theorem and the multiplicative nature of returns, returns from the stock market approximate the lognormal distribution. The lognormal distribution is given by,

$$\mathbf{Z} = \text{Lognormal}\left(\mu, \sigma^2\right) = e^{\mu+\sigma\mathbf{N}(0,1)} \tag{1}$$

where μ and $sigma$ are the mean and standard deviation of the underlying normal distribution, and $\mathbf{N}(0,1)$ is the standard normal, or Gaussian, distribution.

Denote the mean by m, and the standard deviation, or volatility, by s. Here, the mean m denotes the multiplicative return factor, so a 5% increase in value would have the value 1.05.

Assuming lognormality,

$$\begin{aligned} m &= e^{\mu+\frac{\sigma^2}{2}} \\ s &= m\sqrt{e^{\sigma^2}-1} \end{aligned} \tag{2}$$

Two useful facts regarding the lognormal distribution are,

$$k\text{Lognormal}\left(\mu, \sigma^2\right) = \text{Lognormal}\left(\mu + \log k, \sigma^2\right) \tag{3}$$

and if $\mathbf{E}$ is the expectation operator,

$$\begin{aligned} \mathbf{E}\left\langle \text{Lognormal}\left(\mu, \sigma^2\right)^n \right\rangle &= \mathbf{E}\left\langle \text{Lognormal}\left(n\mu, (n\sigma)^2\right)\right\rangle \\ &= e^{n\mu+\frac{n^2\sigma^2}{2}} \end{aligned} \tag{4}$$

both of which can be seen from the definition of the lognormal distribution.

2.2 Geometric Brownian motion

Geometric Brownian motion describes a continuous process that exhibits lognormality over different time scales τ. It is common to define,

$$\alpha = \mu + \frac{\sigma^2}{2} \tag{5}$$

where α is the geometric Brownian motion drift parameter (sometimes confusingly also referred to as μ), so that,

$$m = e^{\alpha} \qquad \text{from 2} \tag{6}$$

Geometric Brownian motion satisfies,

$$\alpha' = \alpha''\tau \tag{7}$$

$$\sigma' = \sigma''\sqrt{\tau} \tag{8}$$

where α' and σ' are the underlying lognormal distribution parameter values for arbitrary timescale τ, and α'' and σ'' are the parameter values for the unit timescale, which is commonly 1 year. These equations can be derived by viewing geometric Brownian motion as the product (or sum in log space) of a large number of independent variables and the central limit theorem.

2.3 Geometric asset combination

Let r be the risk free rate, that is the return on cash, or some other risk free asset, on an annual or other time period basis. In accordance with convention, r is defined as an additive quantity, so a return of 2%, would be represented as the value 0.02.

Let α' and σ' describe describe the risky asset which exhibits geometric Brownian motion. Let π be the risky asset fraction.

It is common to combine the returns of assets additively. This is done when allocations to the asset are set at some point in time and then allowed to wander in accordance with the returns received.

It is less common to combine the returns of assets multiplicatively. This should be done if the allocations to the assets are continuously adjusted in order to maintain a target asset allocation. This is is done here. An easy way of thinking about this is to consider the returns to be an interleaved product, with the risky asset factor being received π of the time, and the risk free factor being received $1-\pi$ of the time. Let μ_v and σ_v describe the portion of the return from the risky, or volatile, asset. The resulting return is then,

$$\begin{aligned}
\mathbf{Z} &= \text{Lognormal}\left(\mu_v, \sigma_v^2\right)(1+r)^{1-\pi} \\
&= \text{Lognormal}\left(\pi\mu', \pi\sigma'^2\right)(1+r)^{1-\pi} && \text{by analogy to 7 and 8} \\
&= \text{Lognormal}\left(\pi\mu' + (1-\pi)\log(1+r), \pi\sigma'^2\right) && \text{by 3}
\end{aligned}$$

which is lognormal. Comparing to equation gives 1,

$$\begin{aligned}
\mu &= \pi\mu' + (1-\pi)log(1+r) \\
\sigma &= \sqrt{\pi}\sigma'
\end{aligned} \tag{9}$$

and thus,

$$\begin{aligned}
\alpha &= \pi\mu' + (1-\pi)log(1+r) + \frac{\pi\sigma'^2}{2} && \text{by 5} \\
&= \pi\alpha' + (1-\pi)log(1+r) && \text{again by 5}
\end{aligned} \tag{10}$$

2.4 Utility

Utility defines the desirability of different levels of consumption C.

Only utility with Constant Relative Risk Aversion (CRRA) will be considered here. This is termed isoelastic utility. It is defined by,

$$U(C) = \frac{C^{1-\gamma}}{1-\gamma} \tag{11}$$

where γ is termed the coefficient of relative risk aversion. The absolute value of utility doesn't matter, only differences in value, so sometimes you will see the above equation written as

$$U(C) = \frac{C^{1-\gamma} - 1}{1-\gamma}$$

so that $U(1) = 0$.

Marginal utility is the incremental value of an incremental unit of consumption,

$$U'(C) = C^{-\gamma} \tag{12}$$

Confusingly, sometimes the symbol γ denotes $1 - \gamma$.

2.5 Notation

In this paper there are three sets of symbols. The symbols adorned by " represent given values for the unit timescale. The symbols adorned by ' represent given values for the τ timescale. And the unadorned symbols represent actual allocations for the τ timescale.

I don't normally speak of r' since,

$$r' = r$$

3 The discrete time period case

The discrete time period version of Merton's portfolio problem is as follows. Over a series of time periods, $t = 0, 1, ...T$, for an investment between a risk free asset with return r and a risky asset that exhibits lognormal returns for each time period described by α' and σ', to determine the continuously reblanced risky allocation fraction π and the consumption amount $c(t)$, so as to maximize expected aggregate CRRA utility of consumption for a coefficient of relative risk aversion γ.

Let $c(t)$ be the fraction of wealth consumed at time t. Wealth starts at some given value $W(0)$ at time $t = 0$ and evolves according to the wealth equation,

$$W(t+1) = W(t)(1 - c(t))\mathbf{Z} \tag{13}$$

for some returns distribution $\mathbf{Z}$.

3.1 Asset allocation

Consider a single time period.

Later, in section 3.2.2 we will see how maximizing utility of terminal wealth for the time peirod is the same as maximizing utility of consumption, but for now take it as given that maximizing utility of terminal wealth is a reasonable thing to want to do.

We seek to maximize expected utility of wealth, $W(t+1)$, for returns distribution $\mathbf{Z}$ given given by equation 1.

$$
\begin{aligned}
\pi &= \underset{\pi}{\operatorname{argmax}} \left\{ \mathbf{E} \left\langle U(W(t+1)) \right\rangle \right\} \\
&= \underset{\pi}{\operatorname{argmax}} \left\{ \mathbf{E} \left\langle \frac{(W(t)(1-c(t))\mathbf{Z})^{1-\gamma}}{1-\gamma} \right\rangle \right\} && \text{by 11 and 13} \\
&= \underset{\pi}{\operatorname{argmax}} \left\{ \frac{e^{(1-\gamma)\mu + \frac{(1-\gamma)^2\sigma^2}{2}}}{1-\gamma} \right\} && \text{by 1 and 4} \\
&= \underset{\pi}{\operatorname{argmax}} \left\{ \frac{e^{(1-\gamma)\alpha - \gamma(1-\gamma)\frac{\sigma^2}{2}}}{1-\gamma} \right\} && \text{by 5} \\
&= \underset{\pi}{\operatorname{argmax}} \left\{ \alpha - \gamma\frac{\sigma^2}{2} \right\} \\
&= \underset{\pi}{\operatorname{argmax}} \left\{ \pi\alpha' + (1-\pi)log(1+r) - \frac{\gamma\pi^2\sigma'^2}{2} \right\} && \text{by 10 and 9}
\end{aligned}
$$

Solving by finding where the derivative with respect to π equals 0 gives,

$$
\begin{aligned}
\alpha' - log(1+r) - \gamma\pi\sigma'^2 = 0 \\
\Rightarrow \pi = \frac{\alpha' - log(1+r)}{\gamma\sigma'^2}
\end{aligned}
\tag{14}
$$

Note that the solution is independent of the time period.

It is important to note that even though utility is assessed once at the end of a time period, the asset allocation normally needs to be continuously maintained over the entire time period at the target asset allocation specified by equation 14. As a result of this continuous rebalancing it is impossible to get "wiped out" despite the possible recommended use of leverage by equation 14. The use of daily resetting leveraged ETFs is closer to what is required than investing once on margin for an entire time period.

3.2 Consumption

This section follows the work of Samuelson.

We seek to determine the optimal consumption amount for over a series of discrete time periods, $t = 0, 1, ...T$. We do this by maximizing the aggregate utility for time periods $t, t+1, ...T$, denoted $J(t, W(t))$, at $t = 0$. J is defined by

$$
J(t, W(t)) = \begin{cases} \mathbf{E} \left\langle U(c(t)W(t)) + J(t+1, W(t+1)) \right\rangle, & t < T \\ \mathbf{E} \left\langle U(c(t)W(t)) \right\rangle, & t = T \end{cases}
\tag{15}
$$

Define the certainty equivalence fraction $b(t)$ by,

$$U(b(t)W(t)) = \frac{J(t, W(t))}{T - t + 1} \tag{16}$$

That is $b(t)$ is the consumption fraction that gives the same expected utility in a single time period as the average expected utility over periods $t, t+1, ...T$.

For the final time period,

$$b(T) = c(T) = 1$$

Suppose $b(t+1)$ known and independent of wealth $W(t+1)$.

For a given $W(t)$ we seek to maximize J as follows,

$$\max \mathbf{E} \langle J(t, W(t)) \rangle$$

$$\Rightarrow \frac{d(\mathbf{E} \langle J(t, W(t)) \rangle)}{dc(t)} = 0$$

$$\Rightarrow \frac{d(\mathbf{E} \langle U(c(t)W(t)) + (T-t)U(b(t+1)W(t)(1-c(t))\mathbf{Z}) \rangle)}{dc(t)} = 0 \qquad \text{by 15, 16, and 13}$$

$$\Rightarrow \mathbf{E} \left\langle U'(c(t)W(t))W(t) - (T-t)U'(b(t+1)W(t)(1-c(t))\mathbf{Z})b(t+1)W(t)\mathbf{Z} \right\rangle = 0$$

Using the definition of marginal utility, equation 12, this implies,

$$c(t)^{-\gamma} = \mathbf{E} \left\langle (T-t)(1-c(t))^{-\gamma} b(t+1)^{1-\gamma} \mathbf{Z}^{1-\gamma} \right\rangle \tag{17}$$

For future use note,

$$\mathbf{E} \left\langle (T-t)(b(t+1)(1-c(t))\mathbf{Z})^{1-\gamma} \right\rangle = (1-c(t))c(t)^{-\gamma} \tag{18}$$

Defining,

$$a = \mathbf{E} \left\langle \mathbf{Z}^{1-\gamma} \right\rangle^{\frac{-1}{\gamma}} \tag{19}$$

Then raising 17 to the power $\frac{-1}{\gamma}$ gives,

$$c(t) = (1-c(t))(T-t)^{\frac{-1}{\gamma}} b(t+1)^{\frac{\gamma-1}{\gamma}} a$$

$$\Rightarrow c(t) = \frac{1}{1 + \frac{1}{(T-t)^{\frac{-1}{\gamma}} b(t+1)^{\frac{\gamma-1}{\gamma}} a}} \tag{20}$$

Now consider J. Using equation 16 and 13, as before, 15 becomes,

$$J(t, W(t)) = \mathbf{E} \langle U(c(t)W(t)) + (T-t)U(b(t+1)W(t)(1-c(t))\mathbf{Z}) \rangle$$

Using the definition of utility 11,

$$J(t, W(t)) = \mathbf{E} \left\langle \frac{((c(t)W(t))^{1-\gamma} + (T-t)(b(t+1)W(t)(1-c(t))\mathbf{Z})^{1-\gamma})}{1-\gamma} \right\rangle$$

By 18,

$$\begin{aligned}
J(t, W(t)) &= \frac{\left(c(t)^{1-\gamma} + (1-c(t))c(t)^{-\gamma}\right) W(t)^{1-\gamma}}{1-\gamma} \\
&= \frac{c(t)^{-\gamma} W(t)^{1-\gamma}}{1-\gamma} \\
&= U\left(c(t)^{\frac{\gamma}{\gamma-1}} W(t)\right) && \text{by 11} \qquad (21) \\
&= U\left(c(t)^{\frac{\gamma}{\gamma-1}} (T-t+1)^{\frac{1}{\gamma-1}} W(t)\right)(T-t+1) && \text{also by 11}
\end{aligned}$$

Comparing this to the definition of the $b(t)$, equation 16, gives,

$$b(t) = c(t)^{\frac{\gamma}{\gamma-1}} (T-t+1)^{\frac{1}{\gamma-1}}$$

which is independent of wealth.

So, according to equation 20,

$$c(t) = \frac{1}{1 + \frac{1}{ac(t+1)}}$$

which is also independent of wealth, as per Samuelson.

Also, as per Samuelson,

$$\begin{aligned}
c(t) &= \frac{ac(t+1)}{1+ac(t+1)} && (22) \\
&= \frac{a^{T-t}}{1+a+a^2+..a^{T-t}} \\
&= \frac{a^{T-t}(a-1)}{a^{T-t+1}-1} && (23)
\end{aligned}$$

Hence the problem is solved, since $\mathbf{E}\left\langle \mathbf{Z}^{1-\gamma} \right\rangle$, and thus a, is determined for the lognormal distribution by equation 4.

3.2.1 Infinite time horizon

In the limit as as $T \to \infty$, from 22 we approach the steady state given by,

$$\begin{aligned}
c &= \frac{ac}{1+ac} \\
\Rightarrow 1 + ac &= a \\
\Rightarrow c &= 1 - \frac{1}{a} && (24)
\end{aligned}$$

3.2.2 Maximizing utility of wealth

By equations 21 and 11,

$$J(t, W(t)) = c(t)^{-\gamma} U(W(t))$$

but for a given t, $c(t)^{-\gamma}$, is a constant. This justifies our maximization of utility of wealth rather than utility of consumption in the single time period case earlier in section 3.1. The consumption derivation did not depend on the asset allocation result, so there is no problem of assuming what is proved.

4 The continuous time case

Suppose the risky asset exhibits geometric Brownian motion in accordance with equations 7 and 8 based on unit time period parameters α'' and σ''. In addition suppose,

$$1 + r = (1 + r'')^{\tau} \tag{25}$$

for the unit timescale risk free rate r''. Then we are concerned with the limits for π and $c(t)$ as $\tau \to 0$.

4.1 Asset allocation

The optimal asset allocation in the continuous time case is given by,

$$\begin{aligned}
\pi &= \frac{\alpha' - \log(1+r)}{\gamma \sigma'^2} && \text{from 14} \\
&= \frac{\alpha'' \tau - \log(1+r'')\tau}{\gamma \sigma''^2 \tau} && \text{by 7, 8, and 25} \\
&= \frac{\alpha'' - \log(1+r'')}{\gamma \sigma''^2}
\end{aligned} \tag{26}$$

4.2 Consumption

First compute a assuming lognormality, and then using equation 23 compute the limit of the consumption rate at time t, $c_{cts}(t)$, as $\tau \to 0$.

$$
\begin{aligned}
a &= \mathbf{E}\left\langle \mathbf{Z}^{1-\gamma}\right\rangle^{\frac{-1}{\gamma}} && \text{from 19} \\
&= e^{\frac{1-\gamma}{-\gamma}\alpha+\frac{1}{2}\frac{(1-\gamma)^2-(1-\gamma)}{-\gamma}\sigma^2} && \text{by 4 and 5} \\
&= e^{(\gamma-1)\left(\frac{\alpha}{\gamma}-\frac{1}{2}\sigma^2\right)} \\
&= e^{(\gamma-1)\left(\frac{\pi\alpha'+(1-\pi)log(1+r)}{\gamma}-\frac{1}{2}\pi^2\sigma'^2\right)} && \text{by 10 and 9} \\
&= e^{(\gamma-1)\left(\frac{1}{2}\pi^2\sigma'^2+\frac{log(1+r)}{\gamma}\right)} && \text{by 14} \\
&= e^{(\gamma-1)\left(\frac{1}{2}\pi^2\sigma''^2+\frac{log(1+r'')}{\gamma}\right)\tau} && \text{by 8 and 25} \\
&= e^{\nu\tau} \text{ where } \nu = (\gamma-1)\left(\frac{1}{2}\frac{(\alpha''-\log(1+r''))^2}{\gamma^2\sigma''^2}+\frac{log(1+r'')}{\gamma}\right) && \text{by 26} \qquad (27)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
c(t) &= \frac{a^{\frac{T-t}{\tau}}(a-1)}{a^{\frac{T-t}{\tau}+1}-1} && \text{from 23} \\
&= \frac{e^{\nu\tau\frac{T-t}{\tau}}\left(1+\nu\tau+o\left(\tau^2\right)-1\right)}{e^{\nu\tau\frac{T-t}{\tau}}e^{\nu\tau}-1} && \text{by 27} \\
&= \frac{\nu\tau e^{\nu(T-t)}+o\left(\tau^2\right)}{\left(e^{\nu(T-t)}\right)(1+o(\tau))-1} \\
&= \frac{\nu\tau}{1-e^{\nu(t-T)}}+o\left(\tau^2\right)
\end{aligned}
$$

Replacing ordinal time t by continuous time $\frac{t}{\tau}$, and single period consumption $c(t)$ by the consumption rate, $c_{cts}(t) = \frac{c(t)}{\tau}$, and taking the limit as $\tau \to 0$,

$$
\begin{aligned}
c_{cts}(t) &= \lim_{\tau\to 0}\frac{c(t)}{\tau} \\
&= \frac{\nu}{1-e^{\nu(t-T)}}
\end{aligned}
$$

As per Merton.

4.2.1 Infinite time horizon

$$
\begin{aligned}
c &= 1-\frac{1}{a} && \text{from 24} \\
&= 1-\frac{1}{1+\nu\tau+o\left(\tau^2\right)} && \text{expanding 27} \\
&= \nu\tau+o\left(\tau^2\right)
\end{aligned}
$$

Replacing single period consumption $c(t)$ by the consumption rate, $c_{cts} = \frac{c(t)}{\tau}$, and taking the limit as $\tau \to 0$,

$$
\begin{aligned}
c_{cts} &= \lim_{\tau \to 0} \frac{c}{\tau} \\
&= \nu
\end{aligned}
$$

5 Conclusion

I have presented a straightforward solution of Merton's portfolio problem for both the discrete and continuous time cases.

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